Spherical beam volume holograms for spectroscopic applications: modeling and implementation

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The spherical beam volume hologram, recorded by a plane wave and a spherical beam, is investigated for spectroscopic applications in detail. It is shown that both the diffracted and the transmitted beam can be used for spectroscopy when the hologram is read with a collimated beam. A new method is introduced and used for analysis of the spherical beam volume hologram that can be extended for analysis of arbitrary holograms. Experimental results are consistent with the theoretical study. It is shown that the spherical beam volume hologram can be used in a compact spectroscopic configuration when the transmitted beam is monitored. Also, on the basis of the properties of the spherical beam hologram, the response of a hologram recorded by a plane wave and an arbitrary pattern is predicted. The information can be used to optimize holographic spectrometer design. © 2004 Optical Society of America *OCIS codes:* 090.7330, 300.6190, 070.2590.

1. Introduction

Sensitive high-resolution compact spectrometers are required for environmental and biosensing. Separation of information of different wavelength channels in spectroscopy requires a dispersive (or wavelengthselective) device, which is implemented by a grating in conventional spectrometers. To avoid ambiguity and irresolvable overlap of different wavelength channels in the output for a spatially incoherent (or diffuse) input, spatial filters are used in conventional spectrometers to limit the angular range of the incident beam. Unfortunately, spatial filtering drastically reduces the photon throughput for diffuse sources. This is a major limitation when conventional spectrometers are used for weak diffuse sources, such as those generated in Raman spectroscopy.

Recently, multimode multiplex spectroscopy was proposed based on use of a weighted projection of multiple wavelength channels (i.e., multimode) of the incident signal to increase the spectrometer sensitivity.¹

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The output signal in multimode multiplex spectroscopy is composed of multiple wavelength channels, and the information of each channel is separated by the postprocessing of the detected signal. The key element in multimode multiplex spectroscopy is a spectral diversity filter (SDF) that maps a homogeneous but diffuse spectral source onto a spatially encoded pattern. Measuring the output light intensity over the output plane by a detector array (for example, a CCD camera) and inverting the spectral–spatial mapping (as outlined in Ref. 1) enables spectral estimation.

Construction of SDFs is constrained by the constant radiance theorem.² According to the constant radiance theorem, it is not possible to produce spatial patterns from a diffuse source without an increase in the mode volume or a reduction in the photon throughput. In contrast with conventional spectroscopy, however, throughput losses with SDFs can be independent of spectral resolution. SDFs have been demonstrated with an inhomogeneous threedimensional photonic crystal.¹ Under the photonic crystal approach, the input-output mode volume is fixed, but a spatially structured fraction of diffuse incident light is reflected. Although threedimensional photonic crystals are attractive as superdispersive elements, they are hard to fabricate based on an arbitrary design. Thus other (more designable and manufacturable) schemes for the development of SDFs are needed.

We recently proposed and demonstrated the feasibility of using spherical beam volume holograms

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(SBVHs) as SDFs.^{3,4} A SBVH is composed of multiple gratings, and the Bragg condition allows for the conversion of a spatially uniform spectrum into a spatial–spectral pattern with a different spatial distribution for different wavelength channels.³ It was shown qualitatively that the spectral diversity performance of these holograms depends on the degree of spatial coherence of the input signal. Although previous results show qualitatively that it is feasible to use SBVHs for SDFs, much effort is needed to optimize the performance of these filters, most probably by means of multiplexing several holograms with optimal patterns. Such an optimization procedure will be more efficient if accurate theoretical models for the performance of these SDFs are available.

In this paper we describe an efficient model for the design and analysis of general volume holograms for SDFs. We first describe, in Section 2, a new theoretical approach that is used to model SBVHs and their diffraction properties. Using this approach, we analyze the major properties of holographic SDFs in Section 3 and compare those results with experimental data in Section 4. Further discussion of these holographic SDFs are presented in Section 5, and final conclusions are made in Section 6.

2. Analysis of Spherical Beam Volume Holograms with the Multigrating Approach

We introduce and implement a new approach to analyze the SBVHs. In general, this approach can be used for any hologram recorded by the interference pattern of an arbitrary coherent beam and a plane wave or even two arbitrary beams. This approach can be used to find the diffracted beam from the hologram when read by a plane wave at any wavelength. The model can be further extended to analyze the case for an arbitrary reading beam. The proof of this approach is presented in Appendix A for a general case. In this section we explain the method for diffraction analysis of a SBVH.

Figure 1 shows general recording and reading setups for SBVHs. The interference pattern of a plane wave and a spherical beam (from a point source) records a SBVH as shown in Fig. 1(a). The recording medium has a thickness of L in the zdirection. It is assumed that the transverse dimensions of the recording material are large compared with L. The point source located at $\mathbf{r}_0 =$ (-a, 0, -d) is formed by a lens with a high numerical aperture. The vector \mathbf{r}_0 makes an angle θ_s with the *z* axis. Therefore *a* is equal to $d \tan(\theta_s)$ in Fig. 1(a). The reference beam is a plane wave with an incident angle θ_r with respect to the z axis. Both recording beams are at wavelength λ with TE polarization (i.e., electric field normal to the incident x-z plane).

To analyze the SBVH recorded in the medium, we first expand the spherical beam at distance $\mathbf{r} = (x, y, y)$



Fig. 1. (a) Recording geometry for a SBVH. The point source is at distance d from the center of the crystal. The reference beam incident angle is θ_r . A line from the coordinate origin to the point source makes an angle θ_s with the z axis. (b) Reading configuration. A collimated beam reads the hologram with a θ'_s incident angle. Note that the direction of the reading beam corresponds to the direction of the signal beam in the recording configuration. The diffracted beam propagates in a direction that makes an angle θ'_r with the z axis. The thickness of the holographic material is L in both cases.

z) from the point source at $\mathbf{r}_0 = (-a, 0, -d)$ as a set of plane waves⁵:

$$\begin{aligned} \frac{1}{|\mathbf{r} - \mathbf{r}_0|} \exp(jk|\mathbf{r} - \mathbf{r}_0|) &= \frac{j}{2\pi} \iint \frac{1}{k_z} \\ &\times \exp[jk_z(z+d)] \\ &\times \exp(jk_x a) \exp[j(k_x x + k_y y)] dk_x dk_y, \end{aligned}$$

where k_x , k_y , and k_z are the x, y, and z components of the wave vector \mathbf{k} , respectively. The magnitude of the wave vector is shown by k. In the expansion of Eq. (1), each component is a plane wave propagating in the direction of unit vector $\hat{\mathbf{a}}_p$ given by

$$\hat{\mathbf{a}}_{p} = \frac{k_{x}}{k}\hat{\mathbf{x}} + \frac{k_{y}}{k}\hat{\mathbf{y}} + \frac{(k^{2} - k_{x}^{2} - k_{y}^{2})^{1/2}}{k}\hat{\mathbf{z}}, \qquad (2)$$

where, in general, $\hat{\mathbf{u}}$ indicates the unit vector in the *u* direction. The constant amplitude and phase of each plane-wave component is given by

$$A(k_x, k_y) = \frac{j}{2\pi (k^2 - k_x^2 - k_y^2)^{1/2}} \exp(jk_x a) \exp(jk_z d) .$$
(3)

Note that in Eq. (1) the integrations are, in general, over all the possible values of k_x and k_y . However, for the values of $|k_x| > k$ or $|k_y| > k$, the *z* component of the propagating vector becomes imaginary, which represents an evanescent wave. The evanescent wave whose amplitude decreases rapidly with *z* can be neglected in the estimation of the integral. Thus the integrals in Eq. (1) essentially take the same values whether they are performed over a circle of radius k (i.e., $k_x^2 + k_y^2 \le k^2$) or over the entire $k_x - k_y$ plane (i.e., from $-\infty$ to $+\infty$). Therefore we omit the range of the integrals in this paper.

The interference of each plane-wave component (traveling in the direction $\hat{\mathbf{a}}_p$) with the reference plane wave records a hologram inside the medium. If we represent the wave-vector components of the spherical beam with (k_x, k_y, k_z) and the incident plane wave with $(k_{rx}, 0, k_{rz}) = [k \sin(\theta_r), 0, k \cos(\theta_r)]$, the effect of the interference pattern on the dielectric constant of the medium can be represented as

$$\varepsilon(r) = \varepsilon_0 + \Delta \varepsilon(k_r, k_v) \exp(j\mathbf{K}_{\sigma} \cdot \mathbf{r}) + \text{c.c.}, \quad (4)$$

where the grating vector \mathbf{K}_{g} is given by

$$\mathbf{K}_{g} = (k_{rx} - k_{x})\mathbf{\hat{x}} + (-k_{y})\mathbf{\hat{y}} + (k_{rz} - k_{z})\mathbf{\hat{z}}.$$
 (5)

The modulation term $\Delta \varepsilon$ is proportional to the amplitudes of the two recording plane waves (the reference beam and a plane-wave component of the signal beam), and therefore it is proportional to $A(k_x, k_y)$. Note that in this analysis we assume that the absorption of the reading beam is weak.

Figure 1(b) shows the reading geometry that is used for holographic SDFs. Note that the reading beam replaces the spherical beam (and not the planewave reference beam). We assume that during readout the hologram is illuminated with an approximately collimated beam at wavelength λ' . The direction of propagation of the reading beam makes an angle θ_s' with the *z* axis as shown in Fig. 1(b). Reading a hologram usually results in both a diffracted beam and a transmitted beam in the output. The main direction of propagation of the diffracted beam makes an angle θ_r' with the *z* axis. In the case of $\lambda = \lambda'$ and $\theta_s = \theta_s'$ (i.e., Bragg-matched readout), the diffracted beam is in the direction of the reference beam as shown in Fig. 1(b), i.e., $\theta_r' = \theta_r$.

The diffracted beam can also be expanded as a sum of plane waves, each corresponding to diffraction of the reading beam by a plane-wave hologram formed by the reference beam and one of the plane-wave components of the recording signal beam described above. To find the diffracted beam from a SBVH, we have to add the diffracted beams from all the different gratings.



Fig. 2. (a) Recording configuration represented in the k domain. The major angular extent of the spherical beam is indicated by $\Delta \theta$ in the k domain. (b) Reading configuration in the k domain. In general, the reading wavelength is different from the recording one. Δk_z is a measure of a partial Bragg-matched condition. All other parameters are the same as those in Fig. 1.

Since the wavelength of the reading beam is, in general, different from the recording wavelength, the dual-wavelength method⁶ should be used to analyze the diffraction from each grating. The *k*-space representations of the recording and reading configurations are shown in Figs. 2(a) and 2(b), respectively. The relations between the Bragg-matched angles (θ_r' and θ_s') and the recording angles (θ_r and θ_s) are⁶

$$\frac{1}{\lambda}\sin\left(\frac{\theta_r+\theta_s}{2}\right) = \frac{1}{\lambda'}\sin\left(\frac{\theta_r'+\theta_s'}{2}\right),\tag{6}$$

$$\theta_r - \theta_s = \theta_r' - \theta_s' \,. \tag{7}$$

Knowing θ_r and θ_s' in Eqs. (6) and (7), the unknowns are θ_s (the signal beam component whose corresponding grating is Bragg matched by the reading beam at θ_s') and θ_r' (the angle of the diffracted beam). When we solve Eqs. (6) and (7), θ_s is found, and therefore the specific grating that is exactly Bragg matched to the collimated reading beam is identified. Other hologram components with grating vectors different from the Bragg-matched one are partially Bragg matched by the reading beam at θ_s' . All these diffractions must be considered to find the complete diffracted beam.

The amplitude of each diffracted beam component is found by use of Born's approximation (as in Ref. 7) with a reading wavelength (in general) different from the recording wavelength. The validity of Born's approximation for these calculations is justified since each plane-wave component causes a low-index modulation. Different components of the diffracted beam will be added to find the total output beam. By use of Born's approximation, the electric field of the diffracted beam from each hologram component is⁷

$$\begin{split} \tilde{E}_{d}(k_{x}, k_{y}, z) &= \frac{j\Delta\varepsilon k'^{2}L}{2\varepsilon_{0}k_{dz'}} \exp[j(K_{gx} + k_{sx}')x] \\ &\times \exp[j(K_{gy} + k_{sy}')y] \exp(jk_{dz}'z) \\ &\times \operatorname{sinc}\!\left[\frac{L}{2\pi}(K_{gz} + k_{sz'} - k_{dz'})\right], \quad (8) \end{split}$$

where the propagation vector of the reading beam is assumed to be $(k_{sx'}, k_{sy'}, k_{sz'})$ with magnitude k', and sinc(u) $\equiv \sin(\pi u)/(\pi u)$. In the configuration shown in Fig. 2(b), the reading beam has a propagation vector of $[(k \sin(\theta_s'), 0, k' \cos(\theta_s')]]$. Note that in general $k' = 2\pi/\lambda'$ is different from $k = 2\pi/\lambda$, where λ and λ' are the recording and reading wavelengths, respectively. The *z* component of the diffracted beam $k_{dz'}$ can be found from

$$k_{dz'} = [k'^2 - (K_{gx} + k_{sx'})^2 - (K_{gy} + k_{sy'})^2]^{1/2}.$$
 (9)

Substituting for K_{gx} from Eq. (5),

$$k_{dz}' = [k'^2 - (k_{rx} + k_{sx}' - k_x)^2 - (k_{sy}' - k_y)^2]^{1/2}.$$
 (10)

Note that each hologram component is represented by one set of (k_x, k_y) in the plane-wave expansion of the recording signal beam.

Combining all the diffraction beam components, the output (diffracted beam) is given by

$$E_d(x, y, z) = \iint \tilde{E}_d(k_x, k_y, z) \mathrm{d}k_x \mathrm{d}k_y \,. \tag{11}$$

Let us define $\tilde{E}_{d'}(k_x, k_y, z)$ as

$$\begin{split} \tilde{E}_{d}'(k_{x}, k_{y}, z) &= 4\pi^{2} \tilde{E}_{d}(k_{x}, k_{y}, z) \exp[j(k_{x}x \\ &+ k_{y}y)] \exp[-j(k_{rx} \\ &+ k_{sx}')x] \exp(-jk_{sy}'y) \\ &= \frac{j2\pi^{2}\Delta\varepsilon k'^{2}L}{\varepsilon_{0}k_{dz}'} \exp(jk_{dz}'z) \\ &\times \operatorname{sinc}\left[\frac{L}{2\pi}(K_{gz} + k_{sz}' - k_{dz}')\right]. \quad (12) \end{split}$$

Then the integral in Eq. (11) can be represented as the inverse Fourier transform of $E_{d}'(k_x, k_y, z)$ as

$$\begin{split} E_d(x, y, z) &= \frac{\exp[j(k_{rx} + k_{sx}')x]\exp(jk_{sy}'y)}{4\pi^2} \\ &\times \iint \tilde{E}_d'(k_x, k_y, z)\exp[-j(k_x x + k_y y)] \\ &\times dk_x dk_y, \end{split}$$
(13)

or

1

$$\begin{split} E_{d}(x, y, z) &= \exp[j(k_{rx} + k_{sx'})x] \exp[jk_{sy'}y] \\ &\times F^{-1} \{E_{d'}(k_{x}, k_{y}, z)\}|_{y \to -y}^{x \to -x}, \end{split} \tag{14}$$

where F^{-1} represents the inverse Fourier operation.

3. Analysis of Spherical Beam Volume Holograms as Spectral Diversity Filters

When a SBVH is read by a collimated beam with a angle $\theta_{s'}$ with respect to the z axis, the diffracted beam can be found when we combine Eqs. (12) and (14). Note that the recorded hologram is represented by the change in the dielectric constant $\Delta \epsilon$. We substitute $\Delta \epsilon$ for the SBVH in Eq. (12) and expand k_z and $k_{dz'}$ in terms of small x and y components of **k** and \mathbf{k}_d using binomial expansion (paraxial approximation). We also assume $z \ll d$ and neglect the small variations for the amplitude because of the $1/k_{dz'}$ term in Eq. (12). All these assumptions are valid for practical implementation of SDFs by use of SBVHs. Using these approximations, we can simplify the output-diffracted beam as

$$\begin{split} E_d(x, y, z) &= C_1 F^{-1} \bigg\{ \exp(jk_x a) \exp\left[-\frac{j}{2k} \right. \\ &\times (k_x^2 + k_x^2) d \left.\right] \operatorname{sinc} \left[\left(K_{gz} + k_{sz}' \right. \\ &\left. - k_{dz}'\right) \frac{L}{2\pi} \right] \bigg\} \bigg|_{\substack{x \to -x, \\ y \to -y}}, \end{split}$$
(15)

where C_1 is a complex constant that includes all terms that do not depend on k_x or k_y . The phase factor outside the integral in Eq. (15) (the phase of C_1) does not affect the spatial intensity distribution of the diffracted beam right after the hologram. Thus we do not explicitly consider it in the rest of our derivation (they are still included in C_1). This closed-form inverse Fourier transform can be found approximately by use of the properties of the Fresnel transform.⁸ For simplicity we show the approach for the case in which the reading beam is normal to the hologram (i.e., $\theta_s' = 0$ or $k_{sx'} = k_{sy'} = 0$). For the more general case, the approach is the same but more algebraic manipulations are needed. Rewriting Eq. (15) in terms of the inverse Fouriertransform integral and representing every parameter in terms of k_x and k_y , we find that

$$\begin{split} E_{d}(x, y, z) &= C_{2} \iint \exp \left\{ -j \, \frac{kd}{2} \left[\left(\frac{k_{x}}{k} - \frac{x-a}{d} \right)^{2} \right. \\ &+ \left(\frac{k_{y}}{k} - \frac{y}{d} \right)^{2} \right] \right\} \\ &\times \operatorname{sinc} \left[\left(K_{gz} + k' - k_{dz}' \right) \frac{L}{2\pi} \right] \frac{\mathrm{d}k_{x}}{k} \frac{\mathrm{d}k_{y}}{k}, \end{split}$$

$$\end{split}$$

$$\end{split}$$

$$(16)$$

where C_2 is another complex constant. The integral in Eq. (16) is the Fresnel transform with parameter $\alpha = kd/2.^8$ For α with a very large absolute value (i.e., $|\alpha| \to \infty$), the Fresnel transform of a function becomes the function itself with a proper change of variable. In Eq. (16), the integrand has a nonzero value for $|k_x/k| \le 1$ and $|k_y/k| \le 1$ and rapidly goes to zero for $|k_x/k| > 1$ or $|k_y/k| > 1$ as discussed in Section 2. Therefore α is very large [typically $d \gg \lambda$ in Fig. 1(a)] compared with the integration variables. Therefore, as an approximate solution, the result of the integral in Eq. (16) is the sinc function with integration variables k_x/k and k_y/k replaced by (x - a)/d and y/d, respectively, i.e.,

$$E_d(x, y, z) \approx C_3 \operatorname{sinc}\left[f\left(\frac{x-a}{d}, \frac{y}{d}\right)\frac{L}{2\pi}\right],$$
 (17)

where again C_3 is another complex constant and the function f(u, v) is

$$f(u, v) = k_z + k' - k(1 - u^2 - v^2)^{1/2} - k \left[\frac{k'^2}{k^2} - \left(\frac{k_{rx}}{k} - u \right)^2 - v^2 \right]^{1/2}.$$
 (18)

For the simple case of $\lambda = \lambda'$, and for $u \le 1$ and $\nu \le 1$, we have

$$f(u, v) \approx \frac{k}{2} \left[1 + \frac{1}{\cos(\theta_r)} \right] \left\{ \left[u - \frac{\sin(\theta_r)}{1 + \cos(\theta_r)} \right]^2 + v^2 - \left[\frac{\sin(\theta_r)}{1 + \cos(\theta_r)} \right]^2 \right\}.$$
(19)

Note that in approximation (19) we used $k_{rx} = k \sin(\theta_r)$ and $k_{rz} = k \cos(\theta_r)$.

It is clear from approximation (17) that the diffracted beam intensity is maximum when the argument of the sinc function [and thus f(u, v)] is zero. The minimum intensity is zero and it occurs when we have

$$f\left(\frac{x-a}{d},\frac{y}{d}\right) = m \frac{2\pi}{L},$$
(20)

with m being a nonzero integer.

From the definition of the function f(u, v), given by Eq. (18), it is clear that the loci of the points with a

constant diffracted intensity (for example, maximum or zero) are a circle. If we consider only the diffracted signal in the main lobe of the sinc function in approximation (17), the diffracted beam will resemble an annulus whose intensity is maximum at the center and goes to zero at the edges at which Eq. (20) holds for $m = \pm 1$.

Figure 3(a) shows the theoretical calculations of the pattern of the diffracted beam of a SBVH recorded with the setup in Fig. 1(a) with d = 1.6 cm and $\lambda = 532$ nm. The holographic material is assumed to have a refractive index of 1.5 and a thickness of 100 μ m. The angles θ_r and θ_s are chosen to be 45° and 0°, respectively. For these calculations, we assumed the dimensions of the holograms in the x and y directions to be 1.5 and 1.5 cm, respectively. A normal incident beam at a 700-nm wavelength reads the hologram. The coordinate origin is shown by O in Fig. 3(a). The dashed box represents the corresponding output region for a hologram with practical dimensions of 3.5 mm \times 3.5 mm. The diffracted pattern from this smaller-size SBVH is shown in Fig. 3(b). It is clear that, because of the smaller size of the hologram, only a portion of the diffracted annulus (which we call a crescent) appears in the output. The existence and some properties of these crescents were experimentally demonstrated recently.^{3,4}

Using approximation (17) and Eq. (20) we can determine several properties of the diffracted crescent. For example, we can calculate the width of the crescent by having y = 0 and finding the distance between the zeros of the main lobe of the sinc function in approximation (17). For the case of identical recording and reading wavelengths, the result is

$$w = \frac{2d\lambda}{L}\cot(\theta_r).$$
 (21)

If we consider the refractive index of the holographic recording material to be n, we can write the width of the crescent as

$$w_{a} = \frac{2\left[d - \frac{L}{2}\left(1 - \frac{1}{n}\right)\right]\lambda_{a}}{L}\cot(\theta_{r,\text{inside}}), \quad (22)$$

where subscript *a* means the parameter measured in the air. The reference angle (θ_r) should be measured inside the material for Eq. (22).

The location of the center of the crescent (maximum intensity) also depends on the reading wavelength λ' . For example, at the y = 0 plane, the crescent is located at

$$x = \frac{a}{d} + \left[\frac{2k_r'(k - k_{rz} - k' + k_r')}{k(k + k_r')} + \frac{k_{rx}^2}{(k + k_r')^2}\right]^{1/2}.$$
(23)

Note that k' and $k_{r'}$ are functions of the reading wavelength λ' . This wavelength dependence of the location of the crescent is the main factor in SBVHs that makes SDFs. Figure 4 shows the diffracted crescent



Fig. 3. (a) Theoretical calculations of the pattern of the diffracted beam of a SBVH recorded with the setup shown in Fig. 1(a) with d = 1.6 cm and $\lambda = 532 \text{ nm}$. The angles θ_r and θ_s are chosen to be 45° and 0°, respectively. The holographic material is assumed to have a refractive index of 1.5 and a thickness of 100 µm. For these calculations, we assumed the dimensions of the holograms in the x and y directions to be 1.5 and 1.5 cm, respectively. The hologram is read by a beam with normal incidence (i.e., propagation along the z axis) at a wavelength of 700 nm. The origin of the coordinate system is shown by O. (b) The diffracted beam pattern of the same SBVH as in (a) but with lateral dimensions of 3.5 mm × 3.5 mm. The corresponding hologram is shown by the dashed box in (a).

calculated with different reading wavelengths of 532, 630, and 700 nm with a normal incident angle. All other parameters are the same as those used for Fig. 3.



Fig. 4. Different crescents for reading with different wavelengths of 532, 630, and 700 nm. All other parameters are the same as those described in the caption of Fig. 3(b).

The wavelength dependency of the location of the crescent can be clearly seen in Fig. 4.

We can also calculate the transmitted beam field pattern (E_t) by subtracting the diffracted field pattern (E_d) from the incident beam pattern $(E_{s'})$, i.e.,

$$\boldsymbol{E}_t = \boldsymbol{E}_{s'} - \boldsymbol{E}_d \,. \tag{24}$$

To calculate the exact value of the field, we can find the inverse Fourier transform in Eq. (14) numerically. We use two-dimensional inverse fast Fourier transform in MATLAB with an adequate sampling rate to verify the approximated approach. We found that the exact numerical results agree well with the approximate results derived above when we simplified Eq. (14). Even with the numerical computation, the method we use to find the diffracted signal is more efficient than the more conventional Born approximation⁹ from a computational point of view. Our method gives the diffracted beam over the desired output plane by calculating only one integral, which can be easily implemented with efficient inverse Fourier-transform techniques such as inverse fast Fourier transform.

4. Experiments

To investigate the properties of the SBVHs for spectroscopy and to check the validity of the theoretical results obtained by the proposed method, we recorded several transmission geometry SBVHs using the setup in Fig. 1(a). The recording material was Aprilis photopolymer.¹⁰ The thicknesses of the samples used were 100, 200, or 300 μ m. The recording wavelength was 532 nm. The values of θ_s and θ_r were -9.6° and 44°, respectively. These angles were selected to allow the operation of the SDF with the normal incident angle at a reading wavelength of around $\lambda' = 800$ nm. The distance of the point source to the hologram (d) varied in the range from 1.6 to 12 cm for different holograms. Both recording beams were TE polarized.

To investigate the performance of SBVHs as SDFs,





(b)

Fig. 5. (a) Diffracted beam from a SBVH illuminated by an approximately collimated white-light beam from the direction of the spherical recording beam. The white light is from a regular 60-W lamp. The white screen is approximately 20 mm from the hologram. The hologram is recorded with the setup shown in Fig. 1(a) with d = 1.6 cm and $\lambda = 532$ nm. The holographic material is Aprilis photopolymer with a refractive index of 1.5 and a thickness of 100 μ m. The angles θ_{e} and θ_{r} in the recording setup are -9.6° and 44°, respectively. (b) The transmitted beam through the SBVH when illuminated by a collimated beam at $\lambda = 700$ nm at a normal incident angle $(\theta'_s = 0^\circ)$. The reading light is obtained when a white-light beam is passed through a monochromator with an output aperture size of 0.45 mm. The full width at halfmaximum of the output spectrum of the monochromator at a 700-nm wavelength is approximately 3 nm. The output of the monochromator is collimated with a collimating lens. The dark $crescent \ in the transmitted beam can be clearly seen. \ The dots in$ the figure correspond to the imperfection in the material.

we read each hologram with reading beams at different wavelengths using the setup in Fig. 1(b). For each reading beam, we monitor both the diffracted beam (diffracted at an angle $\theta_{r'}$ and focused on a screen) and the transmitted beam (at the back face of the hologram using a zoomed CCD camera). We can monitor the spectral diversity of the diffracted beam by reading the hologram with white light. Figure 5(a) shows the diffracted beam from a SBVH that is



Fig. 6. Transmitted beam through the SBVH when read by an approximately collimated white-light beam from the direction of the spherical recording beam. The hologram is the same as that described in the caption of Fig. 5(a).

illuminated by an approximately collimated whitelight beam (from a regular 60-W lamp) from the direction of the spherical recording beam (i.e., $\theta_{s}' \cong$ -10°). The white screen is approximately 20 mm from the hologram. It is clear that different wavelength channels (or colors) of the incident beam are separated at this output plane.

Figure 5(b) shows the transmitted beam through a SBVH when illuminated by a collimated beam at $\lambda' =$ 700 nm at a normal incident angle ($\theta_{s}' = 0^{\circ}$). The incident light is obtained when a white-light beam is passed through a monochromator with an aperture size of 0.45 mm. The full width at half-maximum of the output spectrum of the monochromator at a 700-nm wavelength is approximately 3 nm. The output beam of the monochromator is collimated with a collimating lens. The dark crescent in the transmitted beam resembles the diffracted crescent discussed in Section 3. The shape of this dark crescent is defined by the Bragg selectivity of the SBVH in the x direction in Fig. 1(b). The position of the crescent depends on the incident wavelength and on the incident angle. When the hologram is read with a collimated white-light source, several color crescents appear in the transmitted beam. This is shown in Fig. 6. The color of each crescent corresponds to the

reduction of a diffracted crescent at a specific wavelength from the incident white light.

For quantitative measurements, we define two measures for the dark crescent seen in the output. The first measure is the width of the crescent, which is defined as the distance between the edges of the dark crescent at the back face of the hologram in the *x* direction at y = 0. This measure is directly related to the resolution of the spectrometer. The thinner the crescent, the finer the wavelength resolution of the spectrometer. This measure is the curvature of the crescent. This measure helps us to characterize the expected shape of the detecting signal. It also gives us the information that is useful for the design of rotation-multiplexed spherical beam holograms.³

Figure 7(a) shows the variation of the crescent width with the distance between the point source and the recording material during recording (i.e., d). We obtained the experimental results by recording five holograms at $\lambda = 532$ nm for five different values of d and reading them at both $\lambda' = \lambda = 532$ nm (squares) and $\lambda' = 830$ nm (diamonds). The variations associated with the measurements are also shown as the corresponding error bars. The error bars represent the range of crescent widths measured at different heights of each crescent (i.e., a different value of *y* in Fig. 4) close to the crescent center (y = 0). We also show in Fig. 7(a) the theoretical variations of the crescent width with d, using our theoretical model. The difference between theory and experiment is less than 7%. The limited bandwidth of the reading incident beam (approximately 3 nm FWHM) is the main source of this error. Considering this bandwidth, the theoretical result will be increased approximately 8%, reducing the total difference between the theory and experiment to less than 5%. We used a lens to form the point source of the spherical recording beam. The size of the resulting beam at focus is finite (nonzero). This is an important reason for the difference between the theoretical and experimental results in Fig. 7(a).

As it is clear in Fig. 7(a), the dark crescent becomes wider as *d* increases. To understand this variation, we can use a ray-optics approach¹¹ to relate the coordinates of each point in the hologram to the incident k vectors in the recording spherical beam that originate from the point source. When we increase d, the difference between the k vectors of two fixed points in the hologram plane becomes smaller. On the other hand, the Bragg condition of the hologram allows us for a fixed range of Δk of the original grating vectors to Bragg match an incident collimated beam. Thus, when we increase d, the Bragg-matching region in the k domain (i.e., Δk) corresponds to a larger range in the space domain, resulting in a wider crescent. In the extreme case as $d \to \infty$, the spherical beam becomes a plane wave and the Bragg-matched diffracted beam becomes a plane wave as well, resulting in a dark crescent that is infinitely wide for 100% diffraction efficiency.

Figure 7(b) shows the variation of the crescent



Fig. 7. (a) Variation of the crescent width with the distance between the point source and the recording material during recording [i.e., d in Fig. 1(a)]. Five different holograms are recorded at $\lambda = 532$ nm, each with a different value of *d*. All other recording parameters are the same as those described in the caption of Fig. 5(a). The hologram is read at both $\lambda' = 532$ nm and $\lambda' = 830$ nm. (b) Experimental and theoretical variation of the crescent width with a hologram thickness for 100-, 200-, and 300-µm-thick samples. The recording point source is at a distance of d = 1.6 cm from the hologram for all cases. All other recording parameters are the same as those described in the caption of Fig. 5(a). In both plots, squares and diamonds with the error bars show the experimental results for the reading at 532- and 830-nm wavelengths, respectively. The solid curves show the corresponding theoretical results based on the model described in this paper. In both (a) and (b) the error bars represent the range of crescent widths measured at different heights of each crescent (i.e., different values of y in Fig. 4) close to the crescent center (y = 0).

width with hologram thickness. Again, the experimental results for the reading at $\lambda' = 532$ nm and $\lambda' = 830$ nm are shown with squares and diamonds, respectively, for three different thicknesses. The corresponding error bars as well as the theoretical variations of the width of the crescent as a function of the hologram thickness for reading at 532- and 830-nm wavelengths are also shown. As in Fig. 7(a),

the error bars represent the range of crescent widths measured at different heights of each crescent close to the crescent center (y = 0). The finite bandwidth (approximately 3 nm) of the reading beam is taken into account for these calculations. The agreement between theory and experiment is good, and, on average, the theoretical results are within 10% of the experimental ones. More accurate results are possible with the numerical inverse Fourier transformation as described above. Again, the finite size of the experimental point source mainly contributes to the difference between the theoretical and experimental results. Figure 7(b) shows that thicker holograms result in narrower crescents (i.e., better spectral diversity) with all other parameters fixed. Ŵe explain this by noting that thicker holograms have better wavelength and angular selectivity. Thus the range of grating vectors (i.e., Δk) that diffract the incident collimated beam becomes smaller as the hologram becomes thicker, resulting in a smaller diffracted crescent

The theoretical and experimental shape of the dark crescent read at $\lambda' = 532$ nm and $\lambda' = 830$ nm are depicted in Figs. 8(a) and 8(b), respectively. The reading beam incident angle is approximately 13° for $\lambda' = 532$ nm. The hologram thickness is 300 μ m. All other parameters are the same as those described in the caption of Fig. 5. The agreement between theory and experiment in both cases is good. Note that we assumed that a spherical beam originated from a true point source in our theoretical analysis, which is different from the actual experimental condition. Again, the finite size of the point source in the experiments is the main reason of the difference between the theoretical and experimental results.

5. Discussion

The mapping of different wavelengths to different crescents by SBVHs (as shown in Fig. 4) is useful for the design of compact spectrometers. For these SBVHs, the output signal can be detected at the back face of the hologram, which allows for compact designs. A main limitation of such holographic spectrometers for use with incoherent light is caused by the ambiguity between the wavelength and the angle of the incident beam in the Bragg condition.³ For example, the size of the crescent in Fig. 5(b) becomes larger when the divergence angle of the incident beam increases since the crescents corresponding to different reading plane waves of the same wavelength but different angles of incident occur at different (but close to each other) locations. The same behavior is observed if we keep the incident angle constant but increase the wavelength range of the reading beam. It was shown in Ref. 4 that the acceptable divergence angle for a SBVH spectrometer that can still resolve a dark crescent is 45° in transmission geometry and more than 45° in reflection geometry. One interesting feature of the SBVH is the Fresnel transform relation between the k domain and space domain in these holograms. In conventional plane-wave holograms used in spectroscopy,



 $\lambda = 830 \text{ nm}$

(b)

Fig. 8. Theoretical and experimental shape of the dark crescent in the transmitted beam when the SBVH is read at (a) $\lambda'=532$ nm and (b) $\lambda'=830$ nm. All parameters are the same as those described in the caption of Fig. 5(a).

this relation (k domain to space domain) is governed by a Fourier transformation. Thus decreasing the size of the diffracted beam in one domain results in an increase in the size of that in the other domain. In the Fresnel transform, on the other hand, the quadratic phase factor caused by a spherical recording beam allows for similar variations of size in the two domains. The limitation on this relation is imposed by the distance of the point source to the hologram (d) and the plane in which the dark crescent is observed [L in Fig. 1(a)].

We believe that optimal holographic SDFs must be designed by use of a more complicated spatial profile for the recording beam (compared with a spherical beam). Such a hologram can be recorded by means of interfering a plane wave and a modulated beam obtained when another plane wave is passed through a spatial light modulator (SLM). Having a reliable and efficient simulation tool is essential for the optimization of such holograms. We believe that the method presented here can efficiently be used for such optimization. In analyzing a hologram recorded by a plane wave and a beam from a SLM, we treat each pixel of the SLM as a point source and combine the output crescents corresponding to all these point sources. Since the analysis of the point source (i.e., pixel) can be done by analytic formulation, we can combine this technique with sophisticated optimization schemes such as simulated annealing¹² to find the optimal SLM pattern.

6. Conclusions

We presented a simple and efficient technique for the analysis of diffraction from SBVHs. We showed that the output of a SBVH read by a collimated monochromatic plane wave could be found by use of a Fresnel transform. In special circumstances (which happen in most practical applications), the Fresnel transform can be simplified to an identity transformation by a proper change of variables.

The method presented here can be extended to analyze more complicated holograms when read by plane waves. Although we used the proposed method to analyze holographic SDFs, the method is quite general and can be used for any other application of SBVHs and even more complicated volume holograms. We believe that this method will be useful for the optimization of volume holograms for several applications including spectroscopy.

We used the method for the analysis of SBVH SDFs and showed that the method can predict the experimental results with good accuracy. In particular, we showed that the diffraction of such a SBVH read by a monochromatic plane wave is a circular pattern for a large-size hologram and a crescent-shaped pattern for smaller (practical) holograms. The dependence of the position of this crescent on the reading wavelength in both the diffracted beam and the transmitted beam allows for use of these holograms for spectroscopy.

Appendix A

When a medium with a small perturbation in permittivity $[\Delta \varepsilon(\mathbf{r}')]$ is illuminated by an incident beam \mathbf{E}_p , the diffracted electric field \mathbf{E}_d at position \mathbf{r} is found by use of the Born approximation¹³:

$$\begin{aligned} \mathbf{E}_{d}(\mathbf{r}) &= \frac{1}{4\pi} \int_{V} \frac{\exp(jk|\mathbf{r} - \mathbf{r}'|)}{|\mathbf{r} - \mathbf{r}'|} \nabla' \times \nabla' \\ &\times \left[\frac{\Delta \varepsilon(\mathbf{r}')}{\varepsilon_{0}} \mathbf{E}_{p}(\mathbf{r}') \right] \mathrm{d}v' , \end{aligned} \tag{A1}$$

where the integration is over the volume V, ε_0 is the average (unperturbed) permittivity of the medium, and $\mathbf{r}' = (x', y', z')$ is a position vector in the volume V. In holographic recording, the perturbation in permittivity is caused by the interference between the reference plane wave (\mathbf{E}_r) and the signal beam (\mathbf{E}_s) during the recording. The polarizations of these

fields are considered to be the same for practical cases. In general, the two beams are obtained from a single linearly polarized beam with a beam splitter. Therefore we consider the scalar values of the field $(E_r \text{ and } E_s)$ in our analysis. The perturbation in permittivity in the interference region is

$$\Delta \varepsilon(\mathbf{r}') = \varepsilon_1 E_r(\mathbf{r}') E_s^*(\mathbf{r}') + c.c., \qquad (A2)$$

where ε_1 is a proportional constant, the asterisk (*) shows the complex conjugate operation, and c.c. means the complex conjugate of the preceding term. In the following we consider the first term in Eq. (A2) only because the contribution from the complex conjugate can be found similarly. Suppose that we represent the signal beam (E_s) in a plane parallel to the x'y' plane using its Fourier components as

$$E_{s}(\mathbf{r}') = \iint_{k_{x},k_{y}} A(k_{x}, k_{y}, z') \exp[j(k_{x}x' + k_{y}y')] dk_{x} dk_{y}, \qquad (A3)$$

where the integration is over all values of k_x and k_y . Note that we consider all the factors (such as $1/4\pi^2$ in the inverse Fourier integral) in each component $A(k_x, k_y, z')$.

We substitute $\Delta \varepsilon(\mathbf{r}')$ from Eq. (A2) into Eq. (A1) using the expansion in Eq. (A3). Since the curl operators ($\nabla' \times$) and the integration in Eq. (A1) are functions of \mathbf{r}' only, and the integration in Eq. (A3) is a function of k_x and k_y only, we can change the order of integration over k_x and k_y with the curl operation and integration in Eq. (A1) to obtain

$$\begin{split} \mathbf{E}_{d}(\mathbf{r}) = & \iint_{k_{x} k_{y}} \Biggl\{ \int_{V} \frac{\varepsilon_{1} \exp(jk|\mathbf{r} - \mathbf{r}'|)}{4\pi\varepsilon_{0} |\mathbf{r} - \mathbf{r}'|} \nabla' \times \nabla' \\ & \times \left[A^{*} \left(k_{x}, k_{y}, z' \right) \exp[-j(k_{x}x' + k_{y}y')] \right] \\ & \times E_{r}(\mathbf{r}') \mathbf{E}_{p}(\mathbf{r}') \right] dv' \Biggr\} dk_{x} dk_{y} \,, \end{split}$$
(A4)

where V indicates the whole volume where the hologram is recorded. If the reading beam (\mathbf{E}_p) is a plane wave, the integral over the volume (V) in Eq. (A4) is the diffraction from a simple grating formed by the reference plane wave (E_r) and the plane-wave component from Fourier representation of the signal beam. This diffraction of a plane wave from a simple grating by use of the Born approximation is treated in detail in the literature (for example, see Ref. 7). After we calculate the diffracted plane-wave component $\tilde{\mathbf{E}}(k_x, k_y, z')$ from integration over the holographic volume in Eq. (A4), the total diffracted field is found from

$$\mathbf{E}_{d}(\mathbf{r}) = \iint_{k_{x}k_{y}} \tilde{\mathbf{E}}(k_{x}, k_{y}, z') \mathrm{d}k_{x} \mathrm{d}k_{y}.$$
(A5)

To extend this approach for reading with an arbitrary beam, we can expand the reading beam into its plane-wave components. We find the diffracted beam from each component using Eq. (A5). Then we can find the total diffracted beam by combining all the diffracted components.

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